Introductory Remarks on Linear Algebra
Summer course, 2009, Robert E. Greene

Linear algebra is one of the truly foundational ideas of mathematics, with roots in both basic arithmetic and in Euclidean geometry. The rational numbers are after all a vector space over themselves. And the idea of adding vectors as geometric items goes back to Euclid's construction of a line segment emanating from a given point parallel to and with the same length as a given line segment (Euclid. Elements, Proposition 31, Book I, for the parallelism, the length then being obtained by an obvious laying off of the given length). But linear algebra as a formal subject unto itself is a creation of relatively modern times, arguably beginning with the presentation in 1693 by Cramer of what is today known as Cramer's rule, following the consideration of determinants by Leibnitz. Gaussian elimination was developed by Gauss in the early 19th Century, oddly enough primarily for least squares solution of problems in very applied mathematics, it seems. Linear algebra developed explosively in the mid 19th century in the hands of Cayley, Hamilton, and Sylvester, and others; and by the end of the 19th century it had become an enormous subject. The modern formal concept of a vector space , as opposed to many and various concrete examples, seems to have been introduced by Peano in 1888,and this opened up the possibility of considering linear transformations on vector spaces of infinite dimension: functional analysis was knocking at the door and was soon strongly developed by Hilbert, Banach , and a host of others. In the 20th century the concepts of linear algebra proliferated almost without limit, leading to subjects as various as functional analysis in its present magnificence as well as Lie group theory and representation theory. The growth of the subject in its extended sense continues today without slackening its pace, and for that matter, so does the growth of the subject in its rather more narrow sense. Matrix theory in various forms remains an active research area for example.

No course and certainly not one as brief of this one could begin to discuss even a substantial fraction of the topics that have grown out of linear algebra in the strict sense and that might be considered to be part of linear algebra in the extended sense. (The "Handbook of Banach Spaces ", a survey of the subject, involves two volumes of nearly 1000 pages each!). The intention of this course is only to present those topics that are fundamental for all of the vast array of topics that are part of linear algebra in the broad sense.

Even in this restricted context, there is much to be discussed. And as it happens, there are really two quite distinct ways to discuss the topics. The two ways are associated , roughly speaking, to choosing a basis or not. In the former view, a linear transformation is considered essentially to be a matrix, or more precisely a set of matrices which are related to each other by change of basis for domain and range. In the latter, the linear transformation as an abstraction, as a function from one vector space to another which is linear, is taken as primary. Its matrix representations arise incidentally.

In a sense, these two views are equivalent. But the emphasis is different. The matrix idea is really useful primarily when restricted to finite dimensions, but in that situation if leads to many profound results, especially to the "canonical forms" of one sort and another, results which are less natural in formulation in terms of abstract linear transformations.

The linear transformation idea, on the other hand, is really indispensable in infinite dimensions. Every vector space has a basis (at least in set theory which include the Axiom of Choice), but most naturally arising infinite dimensional vector spaces do not have naturally arising bases. For example, the space of continuous real valued functions on the closed interval [0,1] could hardly be more natural from the viewpoint of modern analysis , but no explicit basis of any sort is known, or even constructible in any reasonable sense.(For one thing, this vector space has uncountable dimension, as we shall see later). And the matrix view of linear transformations of infinite dimensional vector spaces is thus not really natural nor very useful in most instances. (The basis idea is useful in some ways, though. For example, it shows immediately that every vector space has a norm, a fact that is otherwise not at all obvious).

In this course, we are going to divide our attention between the two views, the matrix/basis one and the abstract vector space/linear transformation one. Both are important. And the relationship between them is not hard to see. The dichotomy is not one that is worth being impassioned about!

Determinants were the beginning of the subject, partly because people in those days (the turn of the 17^{th} to the 18^{th} century) people loved formulas. Obsession with determinants for their own sake is largely a thing of the past . Muir's Theory of Determinants in the Historical Order of Development is a monument, but it is a monument to history for the most part. However, the topic remains fundamental. Our text for this course, however, adopts a determinant –free approach, for the most part. So before beginning that text as such, we shall review the basic ideas of determinants separately. (Material will be provided but of course the subject is treated very widely and other material is easily available).

The subject itself at a fairly basic level is divided into two parts: One is relatively close to intuition. A finite dimensional vector space is a linear space with a certain number of "degrees of freedom", its dimension. A linear transformation has a kernel, the dimension of which is the number of degrees of freedom suppressed. Hence the image of the linear transformation has dimension, degrees of freedom, equal to the dimension of the original space – the dimension of the kernel= the original number of degrees of freedom minus the number suppressed by the transformation. This is essentially a physical intuition—which has the fortunate property that in this case it is correct! (One has to be cautious about such things. After all, it is "obvious" that the real line, with one degree of freedom, cannot be spread out continuously to cover the plane, with its two degrees of freedom—obvious but wrong! But such things tend to be true on the linear level. No linear transformation can map the line onto the plane!). Such relatively formalistic developments come quite trippingly off the mathematical tongue—in a sense, much of this early part of the subject could be a series of homework exercises, once the definitions are given. (This does not mean that we shall not discuss them in class ,however—we shall, albeit fairly rapidly).

But other parts of the subject are quite far from being a more or less immediate consequence of the definitions and natural steps from them. It would be hard to argue that Jordan canonical form is an immediate consequence of anything easy! Somewhere in between in terms of depth are the concepts surrounding diagonalization of symmetric matrices. This topic, while not as deep as other canonical form results, occurs so frequently in mathematics as a whole that we shall spend some time exploring it from various viewpoints.

And there are various viewpoints, not only on this but on the subject as a whole, not just the basis-free versus the matrix theory view of the subject but also , and more fundamentally, the question of whether the subject is really part of analysis, with the field involved restricted to the real or complex numbers or whether it is part of algebra, with the field being allowed to be arbitrary.

Certainly linear algebra as algebra has enormous importance. For example, the idea that a field extension E of a ground field F is a vector space over F is a central one in Galois theory. And in any case, in many instance, considering vector spaces over arbitrary fields requires no additional effort.

However, we are going to pursue some aspects of the subject that are specific to the view of it as analysis. First of all, this is interesting—even if one is an algebraist. For example, the fact that one can find an eigenvector for a real symmetric matrix by maximizing a certain function over the unit sphere is irresistibly attractive to anyone who has any feeling for how different aspects of mathematics can relate to each other. And secondly, the relationship between linear algebra and analysis is really important. Finally, I am an analyst, and to some extent this will shape what I tell you.

Of course, the specific content of the course is aimed at preparing the participants for the Basic Examination. That we shall surely accomplish. But in addition, I hope that we shall have time to explore a few byways. Linear algebra is a subject with an amazing depth and variety. So much comes from so little. I hope you will enjoy it as much as I do.

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